Note Title 5/21/2020

Birational Transformations & Minimal Models of Surfaces

Theorem 1 (Elimination of indeterminancy)

$$\phi : \underset{\text{surface}}{X} \dashrightarrow \underset{\substack{\text{projective}\\ \text{variety}}}{Y} \quad \Longrightarrow \quad \begin{tikzcd} & \exists X' \arrow[dl, dotted, "\epsilon_n \cdots \epsilon_1"'] \arrow[dr, "\phi' \text{ morphism}"] \arrow[d, phantom, "\circlearrowright"] & \\ X \arrow[rr, dashed, "\phi"'] & & Y \end{tikzcd}$$

pf: $Y \subseteq \mathbb{P}^m$, H generic hyperplane

WLOG may assume that $Y = \mathbb{P}^m$ and

$\overline{\phi(X)}$ is NOT contained in any hyperplanes.

$\phi^*|H|$ linear system on X

$\vee$

P linear system on X discarding

$|D| \supseteq$ fixed components in $\phi^*|H|$. thus P no fixed component

full linear system

$x \in$ base locus of P finitely many points

$$\begin{tikzcd} \widetilde{X} = Bl_x X \arrow[d, "\epsilon"'] \arrow[dr, dotted, "\tilde{\phi}"] & \\ X \arrow[r, dashed, "\phi"'] & Y \end{tikzcd}$$

$\epsilon^* P \subseteq |\epsilon^* D|$ w/ fixed component kE exceptional divisor

$\vee$ $\quad \widetilde{D} = \epsilon^*D - kE$

$\widetilde{P} \subseteq |\epsilon^* D - kE|$

discarding fixed components from $\epsilon^* P$ again has no fixed component

Notice that $\widetilde{D}^2 = (\epsilon^* D - kE)^2 = D^2 - k^2 < D^2$

Thus, blow up at most D^2 times can eliminate indeterminancy.

Remark: From universal property of blow up

$f^{-1}\mathcal{J}_x \cdot \mathcal{O}_Z$ invertible $\Longrightarrow$ $\exists$

$$\begin{tikzcd} & \widetilde{X} = Bl_x X \arrow[d] \\ Z \arrow[ur] \arrow[r, "f"'] & X \ni x \end{tikzcd}$$

surface

$\Updownarrow$

$f^{-1}(x)$ is a curve

if Z is smooth

Lemma 1: X = irreducible, possibly singular surface

Z = smooth surface

w/ $Z \xrightarrow{f} X \ni x$ birational morphism

s.t f^{-1} undefined at x

$\Longrightarrow$ $f^{-1}(x)$ is a curve

pf: • Cover Z by affine charts and prove the statement w.r.t each chart

$\leadsto$ reduce to the case Z affine. $f^{-1}(p) \neq \phi$

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[d, "f"] & \mathbb{A}^n \\ X \ni x \arrow[u, dotted, bend left, "f^{-1}"] \arrow[ur, dotted] & \end{tikzcd}$$

$j \circ f^{-1} = (x_1, \cdots, x_n)$

$x_1 = \frac{g}{h}$, $h(x) = 0$, g, h coprime

$\therefore$ $f^* g = x_1 \cdot f^* h$ on Z

identify as first component of j

$D = \{f^*h = 0\} \subseteq Z \implies f^*g,\ f^*h$ vanish on D

$\implies D = f^{-1}(\underbrace{\{g = h = 0\}}_{})$ $\Rightarrow x$

g, h coprime

finite points

Remark: In higher dimensional case,

$Z \xrightarrow{f} X$ proper birational, X smooth

$f^{-1}(x)$ is either a point or covered by rational curves.
$\phantom{f^{-1}(}\underset{X}{\cap}$

Theorem 2. If $X \xrightarrow{f} Y$ birational morphism of smooth projective surfaces

then $\exists\ X_n \xrightarrow{\epsilon_n} X_{n-1} \longrightarrow \cdots \xrightarrow{\epsilon_2} X_1 \xrightarrow{\epsilon_1} Y$

sequence of monoidal transformation.

s.t $X \cong X_n$, $f = \epsilon_n \circ \cdots \circ \epsilon_1$.

pf: If $X \cong Y$, then the theorem is true.

Otherwise, f^{-1} undefined at some $y \in Y$

$f^{-1}(y) \subseteq X$ is a curve

$\Longrightarrow$ universal property of blow up

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & \mathrm{Bl}_y Y = X_{n-1} \arrow[ur] & \end{tikzcd}$$

$$f^{-1}(y) \longrightarrow E \longrightarrow y$$

Since $\mathrm{Pic}(X) = \mathrm{Pic}(X_{n-1}) \oplus \mathbb{Z}$

$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*)$ finite rank by long exact sequence of $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$

$$n \leq \mathrm{rk}\,\mathrm{Pic}(X) - \mathrm{rk}\,\mathrm{Pic}(Y)$$

Caveat: It is possible to have infinitely many (-1)-curves. ex generic RES

Theorem 2 motivates the concept of minimal models.

we say $X \geq Y$

Definition: X smooth projective surface $/\mathbb{C}$

X is <u>minimal</u> if $X \longrightarrow Z$ birational, Z smooth, projective then $X \cong Z$

Theorem 2 $\Downarrow$ $\Updownarrow$ $\Uparrow$ Castelnuovo contraction theorem

contains no <u>(-1)-curves</u>

these are the only K_X-negative rational curves

Corollary: X smooth projective surface

then $\exists\, X_{min}$ minimal s.t $X \geq X_{min}$

can be found in smooth category!

Minimal Model Program (Mori, Kawamata, Shokurou
MMP Hacon–McKernon, BCHM, ⋯)

Definition: (X, Δ) is <u>Kawamata log terminal (klt)</u>
can be used for dimension reduction if $\Delta \neq 0$

if
- X normal, $\Delta \geq 0$
- $K_X + \Delta$ $\mathbb{Q}$-Cartier
- $\tilde{X} \xrightarrow{f}$ resolution of singularities

$$K_{\tilde{X}} + f^{-1}_* \Delta = \pi^*(K_X + \Delta) + \sum a_i \underbrace{E_i}_{\text{exceptional divisors}} \quad w/ \quad a_i > -1$$

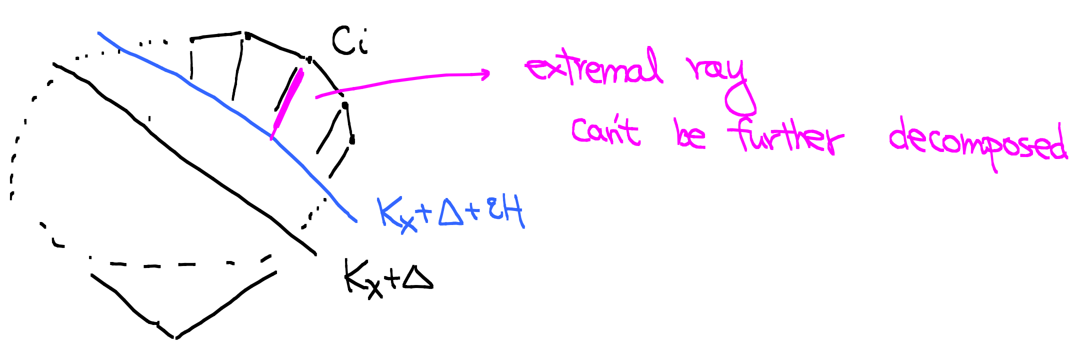
This is the category of singularities s.t the operations of MMP are preserved.

Definition: (X, Δ) is <u>minimal</u> if $K_X + \Delta$ is nef.

(Cone Theorem) (X, Δ) projective klt.

1) $\exists$ countably many rational curves C_i, w/
$$0 < -(K_X + \Delta).C_i < 2\dim X,$$
and
$$\underbrace{\overline{NE(X)}}_{\text{cone of effective curve classes / numerical equivalence}} = \overline{NE(X)}_{-K_X+\Delta \geq 0} + \sum_{\text{countable}} \mathbb{R}[C_i]$$

2) $\forall \varepsilon > 0$, H = ample divisor
$$\overline{NE}(X) = \overline{NE}(X)_{-K_X+\Delta+\varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}[C_i]$$



Contraction theorem

3) $F \subseteq \overline{NE}(X)$ $(K_X+\Delta)$-negative extremal face

$\exists!\ \mathrm{Cont}_F : X \longrightarrow X'$ projective s.t. • $(\mathrm{Cont}_F)_* \mathcal{O}_X = \mathcal{O}_{X'}$ connected fibres

extremal contraction

• $\mathrm{Cont}_F(C) = pt$ iff $[C] \in F$, $C \subseteq X$

MMP procedure

$$(X,\Delta)$$

$$\Big\downarrow \ \text{Cone thm, Contraction thm}$$

$$(X', \Delta' = \mathrm{Cont}_F \Delta) \xrightarrow{\dim X' < \dim X} \text{Mori fibre space}$$

$\dim X = \dim X'$, Cont_F birational

divisorial contraction

exceptional set codim 1

- (X',Δ') stays in category
- nef cone of (X',Δ') smaller
- replace (X,Δ) by (X',Δ')

flipping contraction

exceptional set is codim ≥ 2

- (X',Δ') very singular
- $-(K_X+\Delta)$ is cont_F-ample
- $\exists$

$$\begin{tikzcd}
(X,\Delta) \arrow[rr, dashed, "\text{flip}", "\text{isom in codim 2}"'] \arrow[dr, "\mathrm{Cont}_F"'] & & (X^+,\Delta^+) \arrow[dl, "\mathrm{Cont}_F^+"] \\
& (X',\Delta') &
\end{tikzcd}$$

$\Delta^+ = \Delta$ up to codim 2

Cont_F^+ exceptional set codim ≥ 2

s.t. (X^+,Δ^+) stays in category

$K_{X^+}+\Delta^+$ is cont_F^+-ample

- Replace (X,Δ) by (X^+,Δ^+)

Conjecture: termination of flips?

Theorem (Kleiman's criterion)

X projective, D Cartier, then D ample iff $D > 0$ on $\overline{NE}(X) \setminus \{0\}$.

MMP for surfaces

① $\dim X' = 2$, cont_F is the monoidal transformation.

② $\dim X' = 1$, then X is a ruled surface.

③ $\dim X' = 0$, then $X \cong \mathbb{P}^2$

pf: Assume that C is the contracted rational curve, $K_X \cdot C < 0$

① $C^2 < 0 \xRightarrow[\text{formular}]{\text{adjunction}} C^2 = -1$, cont_F is the monoidal transformation

② $C^2 = 0$, Claim $H^2(X, \mathcal{O}_X(mC)) = 0$, $m \gg 0$

Otherwise $h^0(X, \mathcal{O}_X(mC)) = h^0(X, K_X(-mC))$ but $(K_X - mC) \cdot \underset{\text{ample}}{H} < 0$
Serre duality as $m \gg 0$

Then $H^0(X, \mathcal{O}_X(mC)) \geq \chi(X, \mathcal{O}_X(mC)) = -m\frac{K_X \cdot C}{2} + \chi(\mathcal{O}_X) \geq 2$

$C^2 = 0 \longrightarrow |mC|$ base point free

$\xRightarrow[\text{of contraction}]{\text{uniqueness}} X \xrightarrow[\underset{\mathrm{cont}_C}{\shortparallel}]{|mC|} X'$ curve $\therefore$ X ruled surface

③ $C^2 > 0$, $[C]$ actually falls in the interior of $\overline{NE}(X)$

$+ \mathbb{R}[C]$ extremal ray $\Longrightarrow \rho(X) = 1$

$\xRightarrow[X \text{ projective}]{} C$ ample

$\therefore X \cong \mathbb{P}^2$

ex. $X \hookrightarrow \mathbb{P}^n$ $\quad \{\text{lines in } \mathbb{P}^n\} \cong \mathbb{P}^{n-1}$
$\quad\ \ \rotatebox{90}{$\in$}$
$\quad\ \ x$

$\rightsquigarrow$ projection from x

$$\begin{array}{ccc} X \setminus \{x\} & \longrightarrow & \mathbb{P}^{n-1} \\ \rotatebox{90}{$\in$} & & \\ x' & \longmapsto & \text{line determined by } x, x' \end{array}$$

or

$$\begin{tikzcd} X \arrow[r, dashed] & \mathbb{P}^{n-1} \\ \widetilde{X} = Bl_x X \arrow[u] \arrow[ur] & \end{tikzcd}$$

undefined on x

ex. $X = \underline{\text{quadratic}}$ surface $\subseteq \mathbb{P}^3$

$x \in \quad \rotatebox{90}{$\cong$}$

$\mathbb{P}^1 \times \mathbb{P}^1$

$\rotatebox{90}{$=$}$

$(*x+*y+*z+*w)^2 + (*y+*z+*w)^2 + (*z+*w)^2 + *w^2$

kill $\quad xy, xz, zw \quad\quad yz, yw \quad\quad zw$

up to coordinate change normalized to

$$\{xy - zw = 0\}$$

$$\rightsquigarrow \quad \begin{tikzcd} X \arrow[r, dashed, "\varphi"] & \mathbb{P}^2 \\ \widetilde{X} = Bl_x X \arrow[u, "\in"] \arrow[ur, "f"'] & \end{tikzcd}$$

projection from x

f: blow up of $\mathbb{P}^2$ at $\underline{\text{two points}}$

corresponding to the two lines passing through x.

ex. $p_1, p_2, p_3 \in \mathbb{P}^2$

$$\rightsquigarrow \left\{ \begin{array}{l} \text{linear system of conics} \\ \text{passing throug } p_1, p_2, p_3 \end{array} \right\} =: P \cong \mathbb{P}^2$$

$$\begin{array}{ccc} \mathbb{P}^2 & \overset{\varphi}{\dashrightarrow} & \check{P} \cong \mathbb{P}^2 \\ \Psi & & \Psi \\ p & \longmapsto & (s_0(p), s_1(p), s_2(p)) \end{array}$$

quadratic transformation

$s_i \in P$

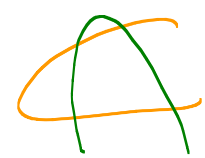
p_1, p_2, p_3 are the base points

$\check{p} \in \check{P}$ corresponding to a pencil in P

$$\{\lambda C^0_{\check{p}} + \mu C^\infty_{\check{p}} = 0\}$$

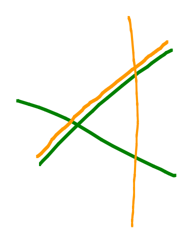
conic conic

- $C^0_{\check{p}} \cap C^\infty_{\check{p}} = 4$ points $= \{p_1, p_2, p_3, p\}$



then $\varphi^{-1}(\check{p}) = p \neq p_1, p_2, p_3$

- $C^0_{\check{p}} \cap C^\infty_{\check{p}} = \underline{\text{line}} \cup \underline{\text{point}}$



the other point

determined by two of p_1, p_2, p_3

WLOG take p_1, p_2, p_3 be $(1:0:0), (0:1:0), (0:0:1)$

then $\varphi(x, y, z) = (x^{-1}, y^{-1}, z^{-1})$

Cremona transformation

Birational Invariants

X: smooth Kähler surface

$q(X) := h^1(X, \mathcal{O}_X)$ irregularity

$P_g(X) := h^2(X, \mathcal{O}_X)$ geometric genus $= P_1(X)$

$P_n(X) := h^0(X, nK_X)$ plurigenera

(Siu) deformation invariants

Proposition 1. The above are birational invariants.

Remark: These are also deformation invariants.

pf: • $q(X) = \frac{1}{2} b_1(X)$. Monoidal transformation doesn't affect $b_1(X)$.

•
$$\begin{tikzcd} X \arrow[r, dashed] & X' \\ U \arrow[u, phantom, "\cup" rotate=90] \arrow[ur, "f"'] & \end{tikzcd}$$

$U = X \setminus \text{pts}$

$$H^0(X', nK_{X'}) \xrightarrow{f^*} H^0(U, nK_U) = H^0(X, K_X)$$

holo. section extends over codim 2 subset.

injection since f^* biholo on a Zariski open subset of X'

$$\Rightarrow \quad P_n(X') \le P_n(X)$$

Exchange X, X', we get $P_n(X') \geq P_n(X)$ $\therefore P_n(X) = P_n(X')$

• X Kähler $\xrightarrow{\quad}$ $h^2(X, \mathcal{O}_X) = h^0(X, K_X)$

Hodge theory

Kodaira Dimension

$$H^0(X, mK_X) \sim m^\kappa + o(m^\kappa).$$

κ is called the Kodaira dimension of X.

Theorem: • κ = dimension of $\mathrm{Im}\left(X \overset{mK_X}{\dashrightarrow} \mathbb{P}^{H^0(mK_X)-1}\right)$, $m \gg 0$

$\cong \mathrm{Proj} \bigoplus_{m\geq 0} H^0(X, mK_X)$

• κ = transcendental degree of $\bigoplus_{m\geq 0} H^0(X, mK_X)$

Q: Is $\bigoplus_{m\geq 0} H^0(X, mK_X)$ finitely generated? Yes Hacon–McKernan.

Proposition 1 $\Longrightarrow$ κ is a birational invariant.

X: smooth projective surface

$\kappa = -\infty$, $P_n \equiv 0 \xrightarrow{\text{Enrique's theorem}} X$ ruled

$\kappa = 0$, $P_n = 0$ or 1. $P_N = 1$ for some N.

Enrique surface, K3 surface, bi-elliptic surface, abelian surface

$\kappa = 1$, $P_n \geq 2$ for some N elliptic surface

$\kappa = 2$, X of general type